

Generalized q -deformed Tamm-Dancoff oscillator algebra and associated coherent states

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In this paper, we propose a full characterization of a generalized q -deformed Tamm-Dancoff oscillator algebra and investigate its main mathematical and physical properties. Specifically, we study its various representations and find the condition satisfied by the deformed q -number to define the algebra structure function. Particular Fock spaces involving finite and infinite dimensions are examined. A deformed calculus is performed as well as a coordinate realization for this algebra. A relevant example is exhibited. Associated coherent states are constructed. Finally, some thermodynamics aspects are computed and discussed.

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I. INTRODUCTION

Quantum algebras and quantum groups play a leading role in physics and mathematics. Quantum groups or q -deformed Lie algebras imply some specific deformations of classical Lie algebras. From the mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo¹ and Drinfeld² (and references therein).

The q -deformation of the oscillator algebra was first accomplished by Arik and Coon³ and lately accomplished by Macfarlane⁴ and Biedenharn⁵ by using the q -calculus which was originally introduced by Jackson in the early 20th century⁶. As matter of other relevant works citation, let us also mention the q -oscillator algebras investigated by Kuryshkin⁷, Jannussis and collaborators⁸, Hounkonnou et al^{9,10} and references therein. In the study of the basic hypergeometric functions, Jackson invented the Jackson derivative and integral, which is now called q -derivative and q -integral. Jackson's pioneering research enabled theoretical physicists and mathematicians to study new physics and mathematics related to the q -calculus. Much was accomplished in this direction and work is under way to find the meaning of the deformed theory.

Historically, the q -deformed Tamm-Dancoff oscillator algebra was first introduced in¹¹, and some of its Hopf algebraic aspects were also discussed in¹². We designate here this model under the name of the TD-oscillator model. It should be pointed out that some of the quantum statistical properties of this model, with the range $q < 1$, have been also considered in^{13,14} in the investigations on the two-parameter-deformed oscillators.

The TD-oscillator model is defined by the following commutation relations

$$aa^\dagger - qa^\dagger a = q^N, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (1)$$

where a , a^\dagger and N are the annihilation, creation and number operators, respectively. The algebra (1) was shown to have a Hopf algebra structure¹². See also¹⁵, where is shown the Hopf algebra structure of a generalized Heisenberg-Weyl algebra.

In this paper, we consider a generalization of the q -deformed Tamm-Dancoff oscillator algebra and investigate its main mathematical and physical properties.

The paper is organized as follows. In Section II, we study the representation for the generalized q -deformed TD oscillator algebra and find the condition satisfied by the deformed

number. In Section III, we find the deformed derivative and deformed integral and obtain a coordinate realization for this algebra. An interesting example is discussed. Associated coherent states are constructed in the Section IV. Section V is devoted to the generalized q -deformed TD oscillator algebra in d -dimensional Fock space. Its representation as well as the deformed derivative and deformed integral are given. Finally, some thermodynamics aspects are discussed in Section VI.

II. GENERALIZATION OF THE Q-DEFORMED TAMM-DANCOFF OSCILLATOR ALGEBRA

Let us consider the following algebra

$$aa^\dagger - a^\dagger a = \{N + 1\} - \{N\}, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (2)$$

where the new q -deformed number is defined as

$$\{N\} = N(\mu q^{\alpha N + \beta} + \eta q^{\gamma N + \delta}), \quad \alpha > 0, \alpha \neq \gamma, q > 0. \quad (3)$$

Meljanac et al¹⁶ introduced the generalized q -deformed single-mode oscillator algebra through the identity operator $\mathbf{1}$, a self-adjoint number operator N , a lowering operator a and an operator \bar{a} which is not necessarily conjugate to a satisfying

$$[N, a] = -a, \quad [N, \bar{a}] = \bar{a}, \quad (4)$$

$$a\bar{a} - F(N)\bar{a}a = G(N), \quad (5)$$

where F and G are arbitrary complex analytic functions. The same algebra was investigated by Bukweli and Hounkonnou¹⁷. These authors show that from the relation (4), one can get

$$[N, a\bar{a}] = 0 = [N, \bar{a}a] \quad (6)$$

implying the existence of a complex analytic function φ such that

$$\bar{a}a = \varphi(N) \quad \text{and} \quad a\bar{a} = \varphi(N + 1). \quad (7)$$

Therefore, (5) can be rewritten as follows:

$$\varphi(N + 1) - F(N)\varphi(N) = G(N), \quad (8)$$

where the structure function $\varphi(n)$ is as follows¹⁷

$$\varphi(n) = [F(n-1)]! \sum_{k=0}^{n-1} \frac{G(k)}{[F(k)]!}, \quad n \geq 1, \quad (9)$$

and

$$[F(k)]! = \begin{cases} F(k)F(k-1) \cdots F(1) & \text{if } k \geq 1 \\ 1 & \text{if } k = 0. \end{cases} \quad (10)$$

Let us denote now a^\dagger the Hermitian conjugate of the operator a . Then,

$$[N, a^\dagger] = a^\dagger \quad \text{and} \quad \bar{a} = c(N)a^\dagger, \quad (11)$$

where $c(N)$ is given by $c(N) = e^{i \arg \varphi(N)}$. The algebra (2) is the particular case of (8) with $\varphi(N) = \{N\}$, $F(N) = q$ and $G(N) = \mu q^{\alpha N + \beta} + \eta q^{\gamma N + \delta}$.

If we choose $\mu = 1, \eta = 0, \alpha = 1, \beta = -1$, the algebra (2) becomes TD-algebra. The algebra (2) is called a generalized q -deformed Tamm-Dancoff oscillator algebra. When q goes to unity, we have $\{N\} = (\mu + \eta)N$. For correspondence, we demand

$$\mu + \eta = 1. \quad (12)$$

Then the relation (3) becomes

$$\{N\}_\mu = N(\mu q^{\alpha N + \beta} + (1 - \mu)q^{\gamma N + \delta}). \quad (13)$$

From now we restrict our concern to the case when q is real. We are interested in the Fock representation of the algebra (2); this is an irreducible representation constructed on a Hilbert space with the orthonormal basis of vectors $|n\rangle$, $n = 0, 1, 2, \dots$. The action of N is standard in the sense that

$$N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \dots, \quad (14)$$

while the action of the remaining operators is given by

$$a|n\rangle = \sqrt{\{n\}_\mu} |n-1\rangle \quad (15)$$

$$a^\dagger|n\rangle = \sqrt{\{n+1\}_\mu} |n+1\rangle. \quad (16)$$

The latter relations require $\{n\}_\mu \geq 0$, implying

$$q^{\alpha n + \beta} \geq \left(1 - \frac{1}{\mu}\right) q^{\gamma n + \delta}. \quad (17)$$

When $\mu \leq 1$, the inequality holds for all n . However, it is not the case when $\mu > 1$. In this case, the solution of the inequality (17) depends on the value of q and $\alpha - \gamma$. Thus, we have the following four cases.

- **Type I :** $q > 1, \alpha > \gamma$

$$n \geq \frac{1}{\alpha - \gamma} \left[\delta - \beta + \log_q \left(1 - \frac{1}{\mu} \right) \right] \quad (18)$$

- **Type II :** $q > 1, \alpha < \gamma$

$$n \leq \frac{1}{\alpha - \gamma} \left[\delta - \beta + \log_q \left(1 - \frac{1}{\mu} \right) \right] \quad (19)$$

- **Type III :** $0 < q < 1, \alpha > \gamma$

$$n \leq \frac{1}{\alpha - \gamma} \left[\delta - \beta + \log_q \left(1 - \frac{1}{\mu} \right) \right] \quad (20)$$

- **Type IV :** $0 < q < 1, \alpha < \gamma$

$$n \geq \frac{1}{\alpha - \gamma} \left[\delta - \beta + \log_q \left(1 - \frac{1}{\mu} \right) \right]. \quad (21)$$

III. THE q -DEFORMED TAMM-DANCOFF OSCILLATOR ALGEBRA WITH AN INFINITE DIMENSIONAL FOCK SPACE

In this section we suggest an interesting example for the algebra (2) with infinite dimensional Fock space. We restrict our concern to the case when $0 < q < 1$. Let us take the following values:

$$\alpha = -1, \quad \beta = 1, \quad \gamma = 1, \quad \delta = -1.$$

With this choice, we have

$$aa^\dagger - a^\dagger a = \{N + 1\}_\mu - \{N\}_\mu, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (22)$$

where

$$a^\dagger a = \{N\}_\mu = N (\mu q^{-N+1} + (1 - \mu)q^{N-1}).$$

This choice gives us an infinite dimensional representation. Therefore, we have

$$N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \dots, \quad (23)$$

$$a|n\rangle = \sqrt{n(\mu q^{-n+1} + (1 - \mu)q^{n-1})} |n - 1\rangle, \quad (24)$$

$$a^\dagger|n\rangle = \sqrt{(n + 1)(\mu q^{-n} + (1 - \mu)q^n)} |n + 1\rangle. \quad (25)$$

In order to have a functional realization of this representation, we consider the space \mathcal{P} of all monomials in variable x , and introduce its basis of monomials

$$|n\rangle := \frac{x^n}{\sqrt{\{n\}_\mu!}}, \quad (26)$$

where

$$\{n\}_\mu! = \prod_{k=1}^n \{k\}_\mu, \quad \{0\}_\mu! = 1. \quad (27)$$

Then, the functional realization of the algebra (22) is given by

$$a := D_x, \quad a^\dagger := x, \quad N := x\partial_x, \quad (28)$$

where the new deformed derivative is given by

$$D_x = (\mu q^{-x\partial_x} + (1-\mu)q^{x\partial_x}) \partial_x = (\mu T_q^{-1} + (1-\mu)T_q) \partial_x, \quad (29)$$

and

$$T_q f(x) = f(qx). \quad (30)$$

The Leibniz rule of the deformed derivative is then given by

$$\begin{aligned} D_x(f(x)g(x)) &= (D_x f(x))(T_q^{-1}g(x)) + (T_q f(x))(D_x g(x)) - \mu(\mathcal{T}f(x))(T_q^{-1}\partial_x g(x)) \\ &\quad + (1-\mu)(T_q \partial_x f(x))(\mathcal{T}g(x)), \end{aligned} \quad (31)$$

where

$$\mathcal{T}f(x) = (T_q - T_q^{-1})f(x) = f(qx) - f(q^{-1}x). \quad (32)$$

Let Q_μ and P_μ be the deformed position and momentum operators defined as follows:

$$Q_\mu := (1/2 m \omega)^{1/2} (a^\dagger + a) \quad \text{and} \quad P_\mu := i (m \omega/2)^{1/2} (a^\dagger - a). \quad (33)$$

The operators $(a^\dagger + a)$ and $i(a^\dagger - a)$ are not essentially self-adjoint, but have a one-parameter family of self-adjoint extensions for $0 < q < 1$.

Indeed, the matrix elements of the operator $a^\dagger + a$ on the basis vector $|n\rangle$ of the space \mathcal{P} are given by

$$\langle m | a^\dagger + a | n \rangle = b_{n,\mu} \delta_{m,n+1} + b_{n-1,\mu} \delta_{m,n-1}, \quad n, m = 0, 1, 2, \dots, \quad (34)$$

while the matrix elements of the operator $i(a^\dagger - a)$ are given by

$$\langle m | i(a^\dagger - a) | n \rangle = i b_{n,\mu} \delta_{m,n+1} - i b_{n-1,\mu} \delta_{m,n-1}, \quad n, m = 0, 1, 2, \dots, \quad (35)$$

where $b_{n,\mu} = \{n+1\}_\mu$. Besides, the operators $(a^\dagger + a)$ and $i(a^\dagger - a)$ can be represented by the two following symmetric Jacobi matrices, respectively,

$$M_{Q_\mu} = \begin{pmatrix} 0 & b_{0,\mu} & 0 & 0 & 0 & 0 & \cdots \\ b_{0,\mu} & 0 & b_{1,\mu} & 0 & 0 & 0 & \cdots \\ 0 & b_{1,\mu} & 0 & b_{2,\mu} & 0 & 0 & \cdots \\ 0 & 0 & b_{2,\mu} & 0 & b_{3,\mu} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (36)$$

and

$$M_{P_\mu} = \begin{pmatrix} 0 & -ib_{0,\mu} & 0 & 0 & 0 & 0 & \cdots \\ ib_{0,\mu} & 0 & -ib_{1,\mu} & 0 & 0 & 0 & \cdots \\ 0 & ib_{1,\mu} & 0 & -ib_{2,\mu} & 0 & 0 & \cdots \\ 0 & 0 & ib_{2,\mu} & 0 & -ib_{3,\mu} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \quad (37)$$

For $0 < q < 1$, we have

$$\lim_{n \rightarrow \infty} b_{n,\mu} = \lim_{n \rightarrow \infty} ((n+1)(\mu q^{-n} + (1-\mu)q^n))^{1/2} = \infty. \quad (38)$$

Considering the series $\sum_{n=0}^{\infty} 1/b_{n,\mu}$, we obtain

$$\overline{\lim}_{n \rightarrow \infty} \left(\frac{1/b_{n+1,\mu}}{1/b_{n,\mu}} \right) = \overline{\lim}_{n \rightarrow \infty} \left(\frac{(n+1)(\mu q^{-n} + (1-\mu)q^n)}{(n+2)(\mu q^{-n-1} + (1-\mu)q^{n+1})} \right)^{1/2} = q^{1/2} < 1. \quad (39)$$

This ratio test leads to the conclusion that the series $\sum_{n=0}^{\infty} 1/b_{n,\mu}$ converges. Moreover, $1 - 2q^2 + q^4 = (1 - q^2)^2 > 0 \implies 2 \leq q^{-2} + q^2$ and $2\mu(1-\mu) \leq \mu(1-\mu)(q^{-2} + q^2) < 2\mu(1-\mu) + \mu^2 q^{-2} + (1-\mu)^2 q^2$. Hence,

$$\begin{aligned} 0 \leq b_{n-1,\mu} b_{n+1,\mu} &= (n^2 + 2n)(\mu^2 q^{-2n} + \mu(1-\mu)(q^2 + q^{-2}) + (1-\mu)^2 q^{2n}) \\ &\leq (n^2 + 2n)(\mu^2 q^{-2n} + \mu(1-\mu) + (1-\mu)^2 q^{2n}) \\ &\quad + (\mu^2 q^{-2n} + \mu(1-\mu) + (1-\mu)^2 q^{2n}) = b_{n,\mu}^2. \end{aligned} \quad (40)$$

Therefore, the Jacobi matrices in (36) and (37) are not self-adjoint (Theorem 1.5., Chapter VII in Ref.¹⁸) but have each a one-parameter family of self-adjoint extensions.

Let

$$H_\mu := \frac{1}{2m}(P_\mu)^2 + \frac{1}{2}m\omega^2(Q_\mu)^2$$

$$= \frac{\omega}{2}(a^\dagger a + aa^\dagger) \quad (41)$$

be the deformed Hamiltonian associated to the algebra (22). The following statement holds:

- The vectors $|n\rangle$ are eigen-vectors of H_μ with respect to the eigenvalues

$$E_\mu(n) = \frac{\omega}{2}(\{n\}_\mu + \{n+1\}_\mu), \quad (42)$$

- The mean values of Q_μ and P_μ in the states $|n\rangle$ are zero while their variances are given by

$$(\Delta Q_\mu)_n^2 = \frac{1}{2m\omega}(\{n\}_\mu + \{n+1\}_\mu) \quad (43)$$

and

$$(\Delta P_\mu)_n^2 = \frac{m\omega}{2}(\{n\}_\mu + \{n+1\}_\mu), \quad (44)$$

respectively, where $(\Delta X)_n^2 = \langle X^2 \rangle_n - \langle X \rangle_n^2$ with $\langle X \rangle_n = \langle n|X|n \rangle$.

- The position-momentum uncertainty relation is given by

$$(\Delta Q_\mu)_n(\Delta P_\mu)_n = \omega^{-1}E_\mu(n), \quad (45)$$

which is reduced, for the vacuum state, to the expression

$$(\Delta Q_\mu)_0(\Delta P_\mu)_0 = \frac{1}{2}. \quad (46)$$

Let us turn back to the derivative (29) and defined the deformed integral as follows:

$$\begin{aligned} \int Dx f(x) &:= \int dx (\mu T_q^{-1} + (1-\mu)T_q)^{-1} f(x) \\ &= \mu^{-1} \sum_{n=0}^{\infty} \left(1 - \frac{1}{\mu}\right)^n \int dx f(q^{2n+1}x). \end{aligned} \quad (47)$$

Applying the deformed derivative and the deformed integral to x^n yields

$$D_x x^n = \{n\}_\mu x^{n-1} \quad \text{and} \quad \int Dx x^n = \frac{x^{n+1}}{\{n+1\}_\mu}.$$

For the deformed derivative and the deformed integral, we have the following formulae

$$\int Dx \frac{1}{x} = \frac{1}{\mu q + (1-\mu)q^{-1}} \ln x \quad \text{and} \quad D_x(\ln x) = \frac{\mu q + (1-\mu)q^{-1}}{x}, \quad (48)$$

where $\int Dx \frac{1}{x}$ exists for $q > \sqrt{1 - \frac{1}{\mu}}$.

The deformed exponential function $\mathcal{E}_\mu(x)$ is defined as

$$\mathcal{E}_\mu(x) := \sum_{n=0}^{\infty} \frac{1}{\{n\}_\mu!} x^n, \quad (49)$$

satisfying

$$D_x \mathcal{E}_\mu(\omega x) = \omega \mathcal{E}_\mu(\omega x) \quad (50)$$

for an arbitrary constant ω . From the relation

$$\int_0^\infty D_x \mathcal{E}_\mu(-\omega x) = \frac{1}{\omega}, \quad (51)$$

we can obtain the following formula

$$\int_0^\infty D_x \mathcal{E}_\mu(-\omega x) x^n = \frac{(-1)^n}{\omega^{n+1}} \prod_{k=1}^n \{-k\}_\mu \quad (52)$$

Inserting $\omega = 1$ into the relation (52) yields

$$\int_0^\infty D_x \mathcal{E}_\mu(-x) x^n = (-1)^n \prod_{k=1}^n \{-k\}_\mu. \quad (53)$$

Replacing μ with $1 - \mu$ in the relation (53), we have

$$\int_0^\infty D_x \mathcal{E}_{1-\mu}(-x) x^n = \frac{2 \{n+2\}_\mu!}{\{2\}_\mu (n+1)(n+2)}, \quad (54)$$

where we use

$$\{-k\}_\mu = -\frac{k}{k+2} \{k+2\}_{1-\mu}. \quad (55)$$

IV. COHERENT STATES

In this section, we construct the coherent states of the generalized TD-oscillator algebra (22). The coherent states $|z\rangle$ are defined as the eigenstates of the annihilation operator in the form

$$a|z\rangle := z|z\rangle. \quad (56)$$

They can be represented by using the eigenvector of the number operator as follows :

$$|z\rangle = \sum_{n=0}^{\infty} c_n(z) |n\rangle. \quad (57)$$

Inserting the relation (57) into (56), we have

$$\sum_{n=1}^{\infty} c_n(z) \sqrt{\{n\}_{\mu}} |n-1\rangle = \sum_{n=0}^{\infty} z c_n(z) |n\rangle. \quad (58)$$

From (58), we get the following recurrence relation

$$c_{n+1}(z) = \frac{z}{\sqrt{\{n+1\}_{\mu}}} c_n(z), \quad n = 0, 1, 2, \dots \quad (59)$$

giving

$$c_n(z) = \frac{z^n}{\{n\}_{\mu}!} c_0(z). \quad (60)$$

From $\langle z|z\rangle = 1$, we have

$$c_0^{-2}(z) = \mathcal{E}_{\mu}(|z|^2) \quad (61)$$

and the coherent states (57) become

$$|z\rangle = \mathcal{E}_{\mu}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\{n\}_{\mu}!}} |n\rangle. \quad (62)$$

They are continuous in their label z . Indeed,

$$\| |z\rangle - |z'\rangle \|^2 = 2(1 - \text{Re}(\langle z|z'\rangle)), \quad (63)$$

where

$$\langle z|z'\rangle = (\mathcal{E}_{\mu}(|z'|^2) \mathcal{E}_{\mu}(|z|^2))^{-1/2} \mathcal{E}_{\mu}(z\bar{z}'). \quad (64)$$

So,

$$\| |z\rangle - |z'\rangle \|^2 \rightarrow 0 \text{ as } |z - z'| \rightarrow 0, \text{ since } \langle z|z'\rangle \rightarrow 1 \text{ as } |z - z'| \rightarrow 0. \quad (65)$$

Besides, their overcompleteness relation can be established as follows:

$$\frac{1}{\pi} \int \int |z\rangle \mu(|z|^2) \langle z| |z| D|z| d\theta = I, \quad (66)$$

where $\mu(|z|^2)$ is a weight function. Inserting (62) into (66), we obtain

$$\sum_{n=0}^{\infty} \frac{1}{\{n\}_{\mu}!} |n\rangle \langle n| \int_0^{\infty} \frac{\mu(x)}{\mathcal{E}_{\mu}(x)} x^n D x = I, \quad (67)$$

where $x = |z|^2$, imposing to find a function f such that

$$\int_0^{\infty} f(x) x^n D x = \{n\}_{\mu}!. \quad (68)$$

Therefore, not all deformed algebras lead to coherent states since the moment problem (68) does not always have solution^{19,20}. So, the question arises is then how to determine the function $f(x)$ in (68)? Using the formula (54), we can write

$$f(x) = \mathcal{E}_{1-\mu}(-x)g(x) = \mathcal{E}_{1-\mu}(-x) \sum_{k=0}^{\infty} g_k x^k. \quad (69)$$

Inserting (69) into (68) yields

$$\sum_{k=0}^{\infty} g_k \frac{(n+k)!}{n!} \prod_{j=n+1}^{n+k+2} \phi(j) = \phi(2), \quad (70)$$

where

$$\phi(j) = \frac{\{j\}_{\mu}}{j}. \quad (71)$$

Solving the above equation, we have

$$g_k = \frac{(-1)^k}{k! \phi(k+2)!} \left[\phi(2) - \sum_{i=0}^{k-1} g_i \frac{(-1)^i k!}{(k-i)!} \prod_{j=k-i+1}^{k+2} \phi(j) \right], \quad k \geq 1$$

with

$$g_0 = 1, \quad \phi(i)! = \prod_{j=1}^i \phi(j). \quad (72)$$

The first few g_k 's are as follows:

$$g_0 = 1 \quad (73)$$

$$g_1 = \frac{\phi(2)(\phi(3) - 1)}{\phi(3)!} \quad (74)$$

$$g_2 = \frac{\phi(2) - \phi(3)\phi(4) + 2\phi(2)(\phi(3) - 1)\phi(4)}{2! \phi(4)!}. \quad (75)$$

On the other way, using the formula (54), we have

$$\phi(2) \int_0^{\infty} Dx \mathcal{E}_{1-\mu}(-x) (\partial_x)^2 x^{n+2} = \{n+2\}_{\mu}!. \quad (76)$$

By replacing $n+2$ by n , the latter equation is equivalent to

$$\int_0^{\infty} Dx (\phi(2)(\partial_x)^2 \mathcal{E}_{1-\mu}(-x)) x^n = \{n\}_{\mu}!, \quad (77)$$

with $\mathcal{E}_{1-\mu}(-x) \partial_x x^n \Big|_0^{\infty} = 0 = \partial_x \mathcal{E}_{1-\mu}(-x) x^n \Big|_0^{\infty}$. Therefore, the function $f(x)$ has the form

$$f(x) = \phi(2) (\partial_x)^2 \mathcal{E}_{1-\mu}(-x) \quad (78)$$

from which the relation

$$\mu(x) = \phi(2) \mathcal{E}_{\mu}(x) (\partial_x)^2 \mathcal{E}_{1-\mu}(-x) \quad (79)$$

follows.

V. d -DIMENSIONAL

In this section we suggest an interesting example for the algebra (3). Now we restrict our concern to the case that $0 < q < 1$. Let us take the following values:

$$\alpha = 1, \beta = -1, \gamma = -1, \delta = -1.$$

We also set

$$\log_q \left(1 - \frac{1}{\mu} \right) = 2d \quad (80)$$

where $d \in Z_+$. In this choice, we have

$$a^\dagger a = \{N\}_d = N \left(\frac{1}{1 - q^{2d}} q^{N-1} - \frac{q^{2d}}{1 - q^{2d}} q^{-N-1} \right). \quad (81)$$

We can easily find that (81) reduces to the Tamm-Dancoff case when d goes to infinity. This choice gives us the d -dimensional representation of algebra (2):

$$N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \dots, d-1 \quad (82)$$

$$a|n\rangle = \sqrt{q^{-1}n \left(\frac{q^n - q^{2d-n}}{1 - q^{2d}} \right)} |n-1\rangle, \quad (83)$$

$$a^\dagger|n\rangle = \sqrt{q^{-1}(n+1) \left(\frac{q^{n+1} - q^{2d-n-1}}{1 - q^{2d}} \right)} |n+1\rangle. \quad (84)$$

With these considerations, the functional realization of the algebra (22) is also given by

$$a := D_x^d, \quad a^\dagger := x, \quad N := x\partial_x, \quad (85)$$

where the deformed derivative D_x^d is given by

$$D_x^d = \frac{q^{x\partial_x} - q^{2d-2-x\partial_x}}{1 - q^{2d}} \partial_x = \frac{T_q - q^{2d-2} T_q^{-1}}{1 - q^{2d}} \partial_x. \quad (86)$$

The Leibniz rule of the deformed derivative is then given by

$$\begin{aligned} D_x^d(f(x)g(x)) &= (D_x^d f(x))(T_q g(x)) + (T_q^{-1} f(x))(D_x^d g(x)) + (\mathcal{T}_d f(x))(T_q \partial_x g(x)) \\ &\quad + q^{2d-2} (T_q^{-1} \partial_x f(x))(\mathcal{T}_d g(x)), \end{aligned} \quad (87)$$

where

$$\mathcal{T}_d f(x) = \frac{T_q - T_q^{-1}}{1 - q^{2d}} f(x) = \frac{f(qx) - f(q^{-1}x)}{1 - q^{2d}}. \quad (88)$$

Let us turn back to the derivative (86) and defined the deformed integral as follows:

$$\begin{aligned}\int D^d x f(x) &= (1 - q^{2d}) \int dx (T_q - q^{2d-2} T_q^{-1})^{-1} f(x) \\ &= q^{2-2d} (q^{2d} - 1) \sum_{n=0}^{\infty} q^{n(2-2d)} \int dx f(q^{2n+1} x).\end{aligned}\quad (89)$$

Applying the deformed derivative and the deformed integral to x^n yields

$$D_x^d x^n = \{n\}_d x^{n-1} \quad \text{and} \quad \int D^d x x^n = \frac{x^{n+1}}{\{n+1\}_d}.$$

As a particular case, we have

$$\int D^d x \frac{1}{x} = q \ln x, \quad D_x^d (\ln x) = \frac{q^{-1}}{x}. \quad (90)$$

The deformed exponential function $E_d(x)$ can be defined as:

$$E_d(x) := \sum_{n=0}^{\infty} \frac{1}{\{n\}_d!} x^n, \quad (91)$$

where

$$\{n\}_d! = \prod_{k=1}^n \{k\}_d, \quad \{0\}_d! = 1 \quad (92)$$

with the property

$$D_x^d E_d(\omega x) = \omega E_d(\omega x) \quad (93)$$

for an arbitrary constant ω . From the relation

$$\int_0^\infty D^d x E_d(-\omega x) = \frac{1}{\omega}, \quad (94)$$

we can derive the following formula

$$\int_0^\infty D^d x E_d(-\omega x) x^n = \frac{(-1)^n}{\omega^{n+1}} \prod_{k=1}^n \{-k\}_d. \quad (95)$$

Setting $\omega = 1$ into the relation (95) yields

$$\int_0^\infty D^d x E_d(-x) x^n = \{n\}_{-d}! \quad (96)$$

where we use

$$\{-k\}_d = -\{k\}_{-d}. \quad (97)$$

VI. DEFORMED BLACK BODY RADIATION

The thermodynamics properties are shown to be determined by the partition function Z defined by

$$Z = \text{Tr}(e^{-\beta H}) = \sum_{n=0}^{\infty} \langle n | e^{-\beta H} | n \rangle, \quad (98)$$

where $\beta = 1/kT$. In the generalized TD (GTD)-oscillator algebra, we assume the Hamiltonian as

$$H := wN. \quad (99)$$

Now we can compute the partition function for the GTD-oscillator as follows:

$$Z = \sum_{n=0}^{\infty} \langle n | e^{-\beta H} | n \rangle = \frac{1}{1 - e^{-\beta w}}. \quad (100)$$

For any operator \hat{O} , the ensemble average is then defined by

$$\langle \hat{O} \rangle := \frac{1}{Z} \text{Tr}(e^{-\beta H} \hat{O}). \quad (101)$$

While the thermodynamics for a system with Hamiltonian (99) is independent of the deformation, Green functions like $\langle a^\dagger a \rangle$ will depend on the deformation. For the symmetric Tamm-Dancoff (STD)-oscillator, we have

$$\langle a^\dagger a \rangle = (e^{\beta w} - 1) \left[\frac{\mu}{(e^{\beta w} - q^{-1})^2} + \frac{1 - \mu}{(e^{\beta w} - q)^2} \right]. \quad (102)$$

As seen in (22) and in the formula (102), this expression of one-particle distribution (also called mean occupation number) separates into two terms by putting $\mu = 0$, or $\mu = 1$, respectively, which are nothing but ordinary Tamm-Dancoff formulae (though with $q \rightarrow 1/q$ in case of $\mu = 1$). On the other hand, the one-particle distribution formula for the usual Tamm-Dancoff model readily follows, at $p \rightarrow q$, of²¹. Besides, when $q \rightarrow 1$, the equation (102) reduces to the classical result of nondeformed case, i.e.,

$$\langle a^\dagger a \rangle = \frac{1}{e^{w/kT} - 1}. \quad (103)$$

It also appears that the mean occupation number for deformed photons obeying the GTD-algebra has a discontinuity at $x = \ln q^{-1}$ for $0 < q < 1$, where $x = \beta w$. Figure 1 shows the discontinuity of $\langle a^\dagger a \rangle$. The discontinuity disappears if we restrict the range of x to $x_{\min} = \ln q^{-1} < x < \infty$. Figure 2 shows the distribution without discontinuity.

Now let us discuss the black body radiation for deformed photons that obey the algebra (22). The mean energy for STD photons is the energy of single GTD photon multiplied by the mean occupation number as follows:

$$\langle E \rangle = w\bar{n} = w(e^{\beta w} - 1) \left[\frac{\mu}{(e^{\beta w} - q^{-1})^2} + \frac{1 - \mu}{(e^{\beta w} - q)^2} \right], \quad (104)$$

where $w = \hbar\nu$ and ν is a frequency of STD photon. The total energy per unit volume for GTD photons in the cavity is obtained by

$$U(T) = \frac{8\pi\hbar}{c^3} \int_0^\infty d\nu \nu^3 (e^{\beta\hbar\nu} - 1) \left[\frac{\mu}{(e^{\beta\hbar\nu} - q^{-1})^2} + \frac{1 - \mu}{(e^{\beta\hbar\nu} - q)^2} \right]. \quad (105)$$

If we set $x = \beta\hbar\nu$, we have

$$U(T) = \frac{8\pi\hbar}{c^3} \left(\frac{kT}{\hbar} \right)^4 \int_0^\infty dx x^3 (e^x - 1) \left[\frac{\mu}{(e^x - q^{-1})^2} + \frac{1 - \mu}{(e^x - q)^2} \right] = a_q T^4, \quad (106)$$

which is the deformed Stefan-Boltzmann law; indeed, the proportional coefficient is q -deformed. Now let us calculate the integral part;

$$J(q) = \int_0^\infty dx x^3 (e^x - 1) \left[\frac{\mu}{(e^x - q^{-1})^2} + \frac{1 - \mu}{(e^x - q)^2} \right] = 12 \sum_{n=0}^\infty \{n\}_\mu \left(\frac{\mu}{(n+1)^4} - \frac{1 - \mu}{(n+2)^4} \right), \quad (107)$$

where we can easily find that $J(q) \rightarrow 6\zeta(4) = \frac{\pi^4}{15}$ when q goes to unity and $\mu = 1/2$ and $\zeta(\cdot)$ is the Riemann zeta function. The infinite series given in (107) diverges. It results from the discontinuity of the mean occupation number of GTD photons. To resolve this problem, we should restrict the range of x to $x_{min} < x$. In this case, the integral part is changed into

$$J(q) = \int_{\ln q^{-1}}^\infty dx x^3 (e^x - 1) \left[\frac{\mu}{(e^x - q^{-1})^2} + \frac{1 - \mu}{(e^x - q)^2} \right]. \quad (108)$$

From the (105), the average energy per mode $I(\nu)$ is given by

$$I(\nu) = \frac{8\pi\hbar\nu^3}{c^3} (e^{\frac{\hbar\nu}{kT}} - 1) \left[\frac{\mu}{(e^{\frac{\hbar\nu}{kT}} - q^{-1})^2} + \frac{1 - \mu}{(e^{\frac{\hbar\nu}{kT}} - q)^2} \right]. \quad (109)$$

Figures 1- 3 show the plots of $I(\nu)$ with $x > x_{min}$ for $q = 0.78, \mu = 0.1, 0.5, 0.9$ (continuous line) and for $q = 1$ (dashed line). One can observe that, for the considered deformation parameter values, the deformed average energy per mode $I(\nu)$ for $q = 0.78$ increases between $\nu = 0$ and $\nu = 4$ while remaining under the values of the non-deformed case ($q = 1$) as μ increases, and maintains the same decreasing trend as the non-deformed case for $\nu \geq 4$.

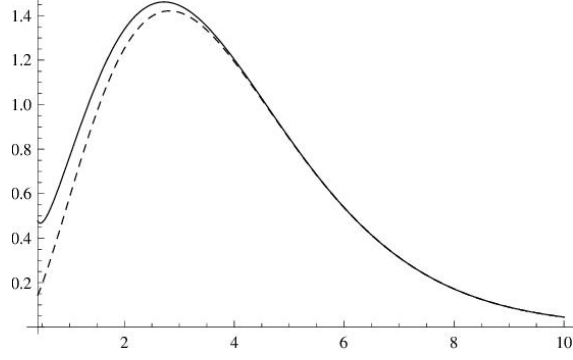


Figure 1. Plot of $I(\nu)$ with $x > x_{min}$ for $q = 0.78, \mu = 0.1$ (continuous line) and for $q = 1$ (dashed line)

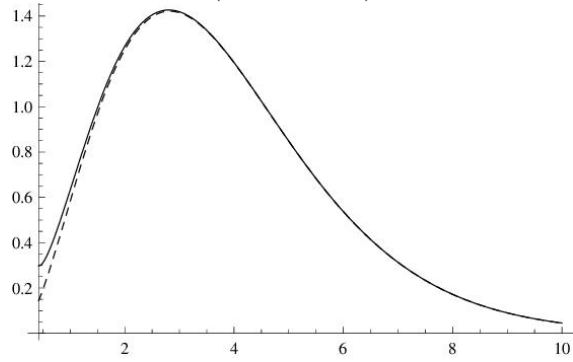


Figure 2. Plot of $I(\nu)$ with $x > x_{min}$ for $q = 0.78, \mu = 0.5$ (continuous line) and for $q = 1$ (dashed line)

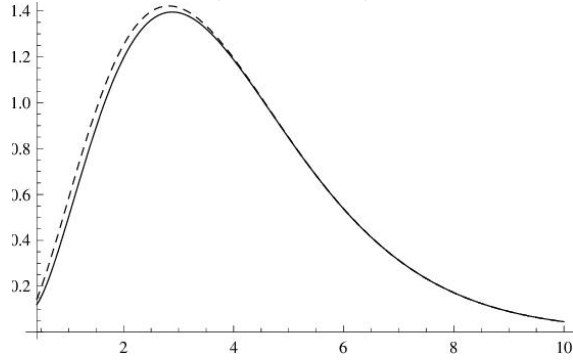


Figure 3. Plot of $I(\nu)$ with $x > x_{min}$ for $q = 0.78, \mu = 0.9$ (continuous line) and for $q = 1$ (dashed line).

VII. CONCLUSION

In this work, we have proposed a full characterization of a generalized TD-oscillator algebra and investigated its main mathematical and physical properties. Specifically, we have studied its various representations and found the condition satisfied by the deformed

q -number to define the algebra structure function. Particular Fock spaces involving finite and infinite dimensions have been examined. A deformed calculus has been performed as well as a coordinate realization for this algebra. A relevant example of the generalized q -deformed TD oscillator algebra has been exhibited. Associated coherent states have been constructed with required mathematical conditions. Besides, some thermodynamics aspects have been computed.

Finally, let us mention that, although the main part of this work dealt with only two parameters q and μ , the investigation of the more general case with a number of parameters greater than two can be performed in a similar way as done in²². For instance, the multi-parameter deformed algebra (2) and its deformed number (3) lead to the following actions for the operators N , a and a^\dagger on the Fock space, for $n = 0, 1, 2, \dots$:

$$N|n\rangle = n|n\rangle, \quad (110)$$

$$a|n\rangle = \sqrt{\{n\}}|n-1\rangle, \quad (111)$$

$$a^\dagger|n\rangle = \sqrt{\{n+1\}}|n+1\rangle, \quad (112)$$

where

$$\{n\} = n(\mu q^{\alpha n + \beta} + \eta q^{\gamma n + \delta}). \quad (113)$$

In this case, the position and momentum operators Q and P ,

$$Q := (1/2 m \omega)^{1/2} (a^\dagger + a) \quad \text{and} \quad P := i (m \omega / 2)^{1/2} (a^\dagger - a), \quad (114)$$

allow us to define the GTD oscillator Hamiltonian operator H and its eigenvalue as:

$$\begin{aligned} H &:= \frac{1}{2m} P^2 + \frac{1}{2} m \omega^2 Q^2 = \frac{\omega}{2} (a^\dagger a + a a^\dagger) \\ &= \frac{\omega}{2} (\{N\} + \{N+1\}) \end{aligned} \quad (115)$$

and

$$E(n) = \frac{\omega}{2} (\{n\} + \{n+1\}), \quad (116)$$

respectively. All other results obtained in this work can be formally extended to the multiparameter deformation case, except for the resolution of the moment problem which can be a difficult task. A thorough analysis of all these questions will be in the core of the forthcoming paper.

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